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LOCAL AND SUPERLINEAR CONVERGENCE OF A CLASS OF VARIABLE METRIC METHODS

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April 1979

(Received February 22, 1979)

Approved for public release
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U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

This paper considers a class of variable metric methods for unconstrained minimization problems. It is shown that with a step size of one each member of this class converges locally and superlinearly.

AMS (MOS) Subject Classification: 90C30

Key Words: Unconstrained minimization, variable metric method, local convergence, superlinear convergence.

Work Unit Number 5 - Mathematical Programming and Operations Research.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

SIGNIFICANCE AND EXPLANATION

Many practical problems in operations research may be reduced to minimizing a function without constraints. Variable metric methods are successfully used in computing a sequence which converges to the minimum of a function. During each iteration a search direction and a step size are computed. In order to obtain fast convergence it is necessary that the chosen step size approximates the optimal step size, i.e., the step size which minimizes the function along the given search direction. This may require considerable computational effort. If an approximation to the minimum is known, however, it is often possible to increase the efficiency of an algorithm by showing that a step size of one is a sufficiently good approximation to the optimal step size. Such a situation arises, for instance, if as in the method of penalty functions a sequence of unconstrained problems is solved in order to obtain a solution to a more complicated optimization problem. In general the minimum of one penalty function is a good approximation to the minimum of the penalty function used next.

In this paper it is shown that variable metric methods converge rapidly with a step size of one if a good approximation of the solution is used as starting point.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

LOCAL AND SUPERLINEAR CONVERGENCE
OF A CLASS OF VARIABLE METRIC METHODS

Klaus Ritter

1. Introduction

If a variable metric method is used to compute a minimizer z of a function $F(x)$ it simultaneously generates a sequence of points $\{x_j\}$ and a sequence of matrices $\{H_j\}$. During each iteration a correction is added to H_j with the intent to construct an approximation to the inverse Hessian matrix of $F(x)$.

A large class of such methods has been introduced by Huang [8]. A restriction of the Huang class to update formulas which are of rank two, satisfy the quasi-Newton equation and maintain the symmetry of H_j leads to a class of methods proposed by Broyden [1] and Fletcher [6]. In [9] global and superlinear convergence has been established for each member of this subclass without the requirement of an optimal step size. However, in the context of a global convergence theory the Broyden-Fletcher-Goldfarb-Shanno-method [2], [6], [7], [10] appears to be the only one for which it can be shown that a step size of one is always acceptable after sufficiently many iterations.

Using a step size of one Broyden, Dennis and Moré [3] have shown that the Broyden-Fletcher-Goldfarb-Shanno-method and the Davidon-Fletcher-Powell-method [4], [5] converge superlinearly to a minimizer z of $F(x)$ provided that the initial point x_0 and the initial matrix H_0 are sufficiently close to z and the inverse Hessian matrix of $F(x)$ at z, respectively. It is the purpose of this paper to extend this result to all members of the Broyden class.

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2. Preliminary results

Let $x \in E^n$ and let $F(x)$ be a real-valued function. If $F(x)$ is twice differentiable at a point x_j we denote the gradient and the Hessian matrix of $F(x)$ at x_j by $g_j = \nabla F(x_j)$ and $G_j = G(x_j)$, respectively. A prime is used for the transpose of a vector or a matrix. For any $x \in E^n$, $\|x\|$ denotes the Euclidean norm of x.

Throughout this paper we require the following assumption to be satisfied.

Assumption 1

There is a vector z such that $F(x)$ is twice continuously differentiable in some convex neighborhood of z, $\nabla F(z) = 0$, $G = G(z)$ is positive definite and the Lipschitz condition

$$\|G(x)-G(z)\| \leq L\|x-z\| \quad , \tag{2.1}$$

where L is a positive constant, is satisfied for all x in some convex neighborhood of z.

Clearly the above assumption implies that there are constants $0 < \mu < \eta$ and a convex neighborhood $U(z)$ such that, for every $x \in U(z)$, the inequality (2.1) and the relation

$$\mu\|y\|^2 \leq y'G(x)y \leq \eta\|y\|^2 \quad \text{for all} \quad y \in E^n$$

hold.

We consider the problem of determining a sequence

$$x_{j+1} = x_j - s_j, \; j = 0,1,2,\ldots \tag{2.2}$$

which converges to z.

If a variable metric method is used to compute the sequence (2.2), then an (n, n)-matrix H_j is associated with each x_j and the search direction s_j is determined by the relation

$$s_j = H_j g_j \quad .$$

The matrix H_{j+1}, associated with x_{j+1}, is obtained by adding a rank two matrix to H_j in such a way that H_{j+1} satisfies the quasi-Newton equation

$$H_{j+1}d_j = p_j \quad ,$$

where

$$d_j = \frac{g_j - g_{j+1}}{\|s_j\|}, \quad p_j = \frac{s_j}{\|s_j\|} .$$

The various variable metric methods differ in the update procedure which is used to compute H_{j+1} from H_j. A large class of such methods has been studied by Broyden [1], Fletcher [6], and Huang [8]. In the following we will consider a subclass of these update procedures which has the property that if the initial matrix H_0 is symmetric all subsequent matrices H_j will be symmetric. It has been shown in [9] that the update formulas that correspond to this subclass can be written in the form

$$(2.3) \qquad H_{j+1} = H_j + \frac{\beta_1(d_j'p_j + d_j'H_jd_j) + \beta_2 d_j'H_jd_j}{d_j'p_j(\beta_1 d_j'p_j + \beta_2 d_j'H_jd_j)} p_jp_j'$$

$$- \beta_1 \frac{p_jd_j'H_j + H_jd_jp_j'}{\beta_1 d_j'p_j + \beta_2 d_j'H_jd_j} - \beta_2 \frac{H_jd_jd_j'H_j}{\beta_1 d_j'p_j + \beta_2 d_j'H_jd_j} ,$$

where β_1 and β_2 are arbitrary parameters with $\beta_1^2 + \beta_2^2 > 0$.

Two well-known members of this class, namely the BFGS - Method (Broyden [2], Fletcher [6], Goldfarb [7], Shanno [10]) and the DFP - Method (Davidon [4], Fletcher, Powell [5]) can be obtained by choosing $\beta_1 = 1, \beta_2 = 0$ and $\beta_1 = 0, \beta_2 = 1$, respectively. The choice $\beta_1 = -\beta_2$ results in the rank one update formula

$$H_{j+1} = H_j + \frac{(p_j - H_jd_j)(p_j' - d_j'H_j)}{(p_j' - d_j'H_j)d_j} .$$

However this method is known to be numerically unstable. It will be excluded in the following.

If H_j is positive definite it has been shown in [9] that H_j can be written in the form

$$(2.4) \qquad H_j = \frac{p_jp_j'}{\rho_j g_j'p_j} + \frac{q_jq_j'}{w_j'q_j} + \sum_{i=3}^{n} \frac{p_{ij}p_{ij}'}{d_{ij}'p_{ij}} ,$$

where

i) $p_j = \frac{H_jg_j}{\|H_jg_j\|}, \quad \rho_j = \frac{1}{\|H_jg_j\|}$,

ii) $w_j \in \text{span}\{g_j, g_{j+1}\}$ such that $w_j'p_j = 0$, $w_j'q_j > 0$ and $q_j = H_jw_j$ has norm one,

iii) the vectors $d_{3j}, \ldots, d_{nj}$ are orthogonal to p_j and q_j and are such that

$$d_{ij}'H_jd_{kj} = 0, \quad i,k = 3,\ldots,n, \quad i \neq k \quad .$$

and

$$p_{ij} = H_jd_{ij}, \quad i = 3,\ldots,n, \quad \text{has norm one.}$$

Then every H_{j+1} determined by (2.3) has the form (see [9])

$$H_{j+1} = \frac{p_jp_j'}{d_j'p_j} + \omega_j \frac{u_ju_j'}{w_j'u_j} + \sum_{i=3}^{n} \frac{p_{ij}p_{ij}'}{d_{ij}'p_{ij}} \quad , \tag{2.5}$$

where the vector u_j is uniquely determined by the conditions

$$u_j \in \text{span}\{q_j, p_j\}, \quad \|u_j\| = 1, \quad d_j'u_j = 0, \quad w_j'u_j > 0 \quad . \tag{2.6}$$

The parameter ω_j depends on the particular numbers β_1 and β_2 used in (2.3). More precisely,

$$\omega_j = \gamma_j \left\| q_j - \frac{d_j'q_j}{d_j'p_j} p_j \right\| \tag{2.7}$$

with

$$\gamma_j = \frac{\beta_1 d_j'p_j + \beta_2 \dfrac{(d_j'p_j)^2}{\rho_j g_j'p_j}}{\beta_1 d_j'p_j + \beta_2 d_j'H_jd_j} \quad . \tag{2.8}$$

This shows that if H_j is positive definite and

$$d_j'p_j = \frac{g_j'p_j - g_{j+1}'p_j}{\|s_j\|} > 0, \quad \text{i.e.,} \quad g_{j+1}'p_j < g_j'p_j$$

then H_{j+1} is positive definite if and only if $\gamma_j > 0$.

3. Superlinear convergence

Throughout this section we will assume that Assumption 1 is satisfied and that the sequence $\{x_j\}$ is generated by the following algorithm.

Algorithm

Step 0: Choose numbers β_1, β_2 with $\beta_1 + \beta_2 \neq 0$, a vector x_0, and a symmetric positive definite matrix H_0. Compute $g_0 = \nabla F(x_0)$. If $g_0 = 0$, stop; otherwise set $j = 0$ and go to Step 1.

Step 1: Set

$$s_j = H_j g_j \quad \text{and} \quad x_{j+1} = x_j - s_j \quad .$$

Compute $g_{j+1} = \nabla F(x_{j+1})$. If $g_{j+1} = 0$, stop; otherwise go to Step 2.

Step 2: Compute H_{j+1} by (2.3). Replace j with $j+1$ and go to Step 1.

We will show that for every choice of β_1 and β_2 with $\beta_1 + \beta_2 \neq 0$ there are numbers

$$\delta = \delta(\beta_1, \beta_2) \quad \text{and} \quad \delta^* = \delta^*(\beta_1, \beta_2)$$

such that

$$\|H_0 - G^{-1}\| \leq \delta \quad \text{and} \quad \|x_0 - z\| \leq \delta^*$$

imply that every H_j is a well-defined positive definite matrix and that the algorithm either terminates after a finite number of iterations or generates a sequence $\{x_j\}$ which converges superlinearly to z.

The convergence proof is based on an estimate for the trace of the matrix

$$G^{1/2} H_j G^{1/2} + G^{-1/2} B_j G^{-1/2} \tag{3.1}$$

where $G^{1/2}$ is the square root of the positive definite matrix G, $G^{-1/2} = (G^{1/2})^{-1}$ and $B_j = H_j^{-1}$. It follows immediately from (2.4) and (2.5) that

$$B_j = \frac{\rho_j g_j g_j'}{g_j' p_j} + \frac{w_j w_j'}{w_j' q_j} + \sum_{i=3}^{n} \frac{d_{ij} d_{ij}'}{d_{ij}' p_{ij}} \tag{3.2}$$

and

$$B_{j+1} = \frac{d_j d_j'}{d_j' p_j} + \frac{1}{\omega_j} \frac{w_j w_j'}{w_j' u_j} + \sum_{i=3}^{n} \frac{d_{ij} d_{ij}'}{d_{ij}' p_{ij}} \; . \tag{3.3}$$

Observe that by (2.6)

$$u_j = \frac{q_j + \alpha_j p_j}{\|q_j + {}_j p_j\|}, \quad \alpha_j = \frac{-d_j' q_j}{d_j' p_j} \; .$$

Therefore, we have $\|q_j + \alpha_j p_j\| w_j' u_j = w_j' q_j$ and by (2.7)

$$\omega_j \frac{u_j u_j'}{w_j' u_j} = \gamma_j \frac{(q_j + \alpha_j p_j)(q_j + \alpha_j p_j)'}{w_j' q_j} \tag{3.4}$$

$$\frac{1}{\omega_j} \frac{w_j w_j'}{w_j u_j} = \frac{1}{\gamma_j} \frac{w_j w_j'}{w_j' q_j} \; . \tag{3.5}$$

Let ψ_j denote the trace of the matrix (3.1). Since the trace of a matrix is equal to the sum of its diagonal elements it follows from (2.4), (2.5) and (3.2) through (3.5) that we have the following relation between ψ_{j+1} and ψ_j.

$$\begin{aligned} \psi_{j+1} = \psi_j &- \frac{p_j' G p_j + \rho_j^2 g_j' G^{-1} g_j}{\rho_j g_j' p_j} + \frac{p_j' G p_j + d_j' G^{-1} d_j}{d_j' p_j} \\ &- \frac{q_j' G q_j + w_j' G^{-1} w_j}{w_j' q_j} + \frac{1}{\gamma_j} \frac{\gamma_j^2 (q_j + \alpha_j p_j)' G (q_j + \alpha_j p_j) + w_j' G^{-1} w_j}{w_j' q_j} \; . \\ = \psi_j &+ \left(\frac{p_j' G p_j + d_j' G^{-1} d_j}{d_j' p_j} - 2 \right) \\ &+ (\gamma_j - 1) \frac{q_j' G q_j}{w_j' q_j} + \left(\frac{1}{\gamma_j} - 1 \right) \frac{w_j' G^{-1} w_j}{w_j' q_j} - \left(\frac{p_j' G p_j + \rho_j^2 g_j' G^{-1} g_j}{\rho_j g_j' p_j} - 2 \right) \\ &+ \gamma_j \left(\frac{(q_j + \alpha_j p_j)' G (q_j + \alpha_j p_j)}{w_j' q_j} - \frac{q_j' G q_j}{w_j' q_j} \right) \; . \end{aligned} \tag{3.6}$$

In the following five lemmas we will establish some properties of ψ_j and the terms on the right hand side of (3.6) which will enable us to prove the key result that the sequence $\{\psi_j\}$ is bounded.

Lemma 1

Let H_j be positive definite. Then

i) $\psi_j \geq 2n$

ii) $\|H_j\| \leq \psi_j \|G^{-1}\|$ and $\|H_j^{-1}\| \leq \psi_j \|G\|$.

Proof:

Let $x, y \in E^n$ be such that $y'x \neq 0$. Set $v = y-Gx$. Then (see [9]),

$$\frac{x'Gx+y'G^{-1}y}{y'x} = 2 + \frac{v'G^{-1}v}{y'x} \quad . \tag{3.7}$$

The first statement of the lemma follows immediately from this equality. By definition ψ_j is equal to the sum of the eigenvalues of the two matrices $G^{1/2}H_jG^{1/2}$ and $G^{-1/2}H_j^{-1}G^{-1/2}$. Since both matrices are positive definite we have

$$\|G^{1/2}H_jG^{1/2}\| \leq \psi_j \quad \text{and} \quad \|G^{-1/2}H_j^{-1}G^{-1/2}\| \leq \psi_j \quad .$$

This comples the proof of the lemma.

Lemma 2

Let H_j be positive definite. For every $0 < \lambda < 1$ there are constants $\tau_1 > 2n$ and $\tau_1^* > 0$ such that, for every j,

$$\psi_j \leq \tau_1 \quad \text{and} \quad \|x_j - z\| \leq \tau_1^*$$

imply

$$\|x_{j+1}-z\| \leq \lambda \|x_j - z\| \quad . \tag{3.8}$$

Proof:

Because

$$\frac{p_j'Gp_j+\rho_j^2q_j'G^{-1}q_j}{\rho_j'q_j'p_j} - 2 \leq \psi_j - 2n$$

it follows from (3.7) and Lemma 1 that

$$\|\rho_jq_j-Gp_j\|^2 = O(\psi_j-2n) \quad \text{and} \quad \|s_j\| = O(\|q_j\|) \quad . \tag{3.9}$$

Therefore, x_j and x_{j+1} are in $U(z)$ for τ_1 and τ_1^* sufficiently small and we obtain from Taylor's theorem the relation

$$g_{j+1} = g_j - Gs_j - E_j s_j \tag{3.10}$$

where

$$E_j = \int_0^1 G(x_j - ts_j)dt - G$$

and

$$\|E_j\| \le \max_{0\le t\le 1} \|G(x_j - ts_j) - G\| \tag{3.11}$$

$$\le \max_{0\le t\le 1} \|L(x_j - t(x_j - x_{j+1}) - z)\|$$

$$\le L \max\{\|x_j - z\|, \|x_{j+1} - z\|\} \ .$$

Therefore,

$$\frac{\|g_{j+1}\|}{\|g_j\|} \le \left\| \frac{g_j}{\|g_j\|} - G \frac{s_j}{\|g_j\|} \right\| + \|E_j\| \frac{\|s_j\|}{\|g_j\|} \tag{3.12}$$

$$= \frac{\|s_j\|}{\|g_j\|} \|\rho_j g_j - Gp_j\| + \|E_j\| \frac{\|s_j\|}{\|g_j\|} \ .$$

Since Taylor's theorem and Assumption 1 imply

$$\|g_j\| = O(\|x_j - z\|) \quad \text{and} \quad \|x_j - z\| = O(\|g_j\|) \ , \tag{3.13}$$

it follows from (3.9), (3.11) and (3.12) that $\|x_{j+1} - z\| \le \lambda\|x_j - z\|$ for τ_1 and τ_1^* sufficiently small.

Lemma 3

Let H_j be positive definite. Then there are constants $2n < \tau_2 \le \tau_1$ and $0 < \tau_2^* \le \tau_1^*$ such that, for every j,

$$\psi_j \le \tau_2 \quad \text{and} \quad \|x_j - z\| \le \tau_2^*$$

imply

i) H_{j+1} is well-defined and positive definite.

ii) $|1-\gamma_j| = O\left(\frac{p_j'Gp_j - \rho_j^2 q_j' G^{-1} q_j}{\rho_j q_j' p_j} - 2\right) + O(\|x_j - z\|)$.

Proof:

By (2.5) and (2.7), H_{j+1} is well-defined and positive definite if $d_j'p_j > 0$ and $\gamma_j > 0$. Using (3.10) we obtain

$$d_j'p_j = p_j'Gp_j + p_j'E_jp_j \quad .$$

Since by (3.8) and (3.11)

(3.14) $$\|E_j\| = O(\|x_j - z\|)$$

this shows that $d_j'p_j > 0$ for τ_2 and τ_2^* sufficiently small.

By (2.4)

$$d_j'H_jd_j = \frac{(d_j'p_j)^2}{\rho_j q_j' p_j} + \frac{(d_j'q_j)^2}{w_j'q_j} \quad .$$

Therefore it follows from (2.8) that

(3.15) $$1 - \gamma_j = \frac{(d_j'q_j)^2}{w_j'q_jd_j'p_j}\left[\beta_1 + \beta_2 \frac{d_j'p_j}{\rho_j q_j'p_j} + \beta_2 \frac{(d_j'q_j)^2}{w_j'q_jd_j'p_j}\right]^{-1} \quad .$$

By definition,

(3.16) $$|d_j'q_j| = \left|\frac{q_j'q_j - q_{j+1}'q_j}{\|s_j\|}\right| = \frac{|q_{j+1}'q_j|}{\|s_j\|} \leq \frac{\|q_j\|}{\|s_j\|}\,\frac{\|q_{j+1}\|}{\|q_j\|} \quad ,$$

(3.17) $$\left|\frac{d_j'p_j}{\rho_j q_j' p_j} - 1\right| = \left|\frac{q_j'p_j - q_{j+1}'p_j}{q_j'p_j} - 1\right| \leq \frac{\|q_j\|}{q_j'p_j}\,\frac{\|q_{j+1}\|}{\|q_j\|} \quad .$$

It follows from part ii) of Lemma 1 that the sequences $\{1/w_j'q_j\}$, $\{1/d_j'p_j\}$, $\{\|q_j\|/\|s_j\|\}$ and $\{\|q_j\|/q_j'p_j\}$ are bounded. Because $\beta_1 + \beta_2 \neq 0$ we deduce, therefore, from (3.9), (3.12), and (3.14) through (3.17) that

(3.18) $$|1-\gamma_j| = O\left(\frac{\|g_{j+1}\|^2}{\|g_j\|^2}\right) < 1$$

for τ_2 and τ_2^* sufficiently small. Furthermore, using (3.18) and (3.12) we obtain

$$|1-\gamma_j| = O(\|\rho_j g_j - Gp_j\|^2 + \|E_j\|^2)$$

which by (3.7) and (3.14) implies the second part of the lemma.

Lemma 4

Let H_j be positive definite and $0 < t < 1$. There are constants $2n < \tau_3 \leq \tau_2$, $0 < \tau_3^* \leq \tau_2^*$ and $\delta_1 > 0$ such that, for every j,

$$\psi_j \leq \tau_3 \quad \text{and} \quad \|x_j - z\| \leq \tau_3^*$$

imply

i) $$\frac{p_j' G p_j + d_j' G^{-1} d_j}{d_j' p_j} - 2 \leq \delta_1 \|x_j - z\|$$

ii) $$\gamma_j \frac{(q_j + \alpha_j p_j)' G (q_j + \alpha_j p_j) - q_j' G q_j}{w_j' q_j} \leq \delta_1 \|x_j - z\|$$

iii) $$(\gamma_j - 1)\frac{q_j' G q_j}{w_j' q_j} + \left(\frac{1}{\gamma_j} - 1\right)\frac{w_j' G^{-1} w_j}{w_j' q_j}$$

$$- t\left(\frac{p_j' G p_j + \rho_j^2 g_j' G^{-1} g_j}{\rho_j g_j' p_j} - 2\right) \leq \delta_1 \|x_j - z\| .$$

Proof:

It follows from (3.10) that

$$d_j - Gp_j = E_j p_j$$

which by (3.7) and (3.14) implies the first part of the lemma. In order to prove the second part we observe that

$$(q_j+\alpha_j p_j)'G(q_j+\alpha_j p_j) - q_j'Gq_j = 2\alpha_j p_j'Gq_j + \alpha_j^2 p_j'Gp_j$$

$$= -2\frac{d_j'q_j}{d_j'p_j}(d_j'-p_j'E_j)q_j + \left(\frac{d_j'q_j}{d_j'p_j}\right)^2 p_j'(d_j-E_jp_j)$$

$$= -\frac{(d_j'q_j)^2}{d_j'p_j} + 2\frac{d_j'q_j}{d_j'p_j}p_j'E_jq_j - \left(\frac{d_j'q_j}{d_j'p_j}\right)^2 p_j'E_jp_j$$

$$= O(\|E_j\|) \quad .$$

Because of (3.18) and (3.14) this equality implies the second part of the lemma for τ_3 and τ_3^* sufficiently small. Finally we have

$$(\gamma_j - 1)\frac{q_j'Gq_j}{w_j'q_j} + \frac{1-\gamma_j}{\gamma_j}\frac{w_j'G^{-1}w_j}{w_j'q_j} = \tag{3.19}$$

$$(\gamma_j - 1)\left[\frac{\gamma_j-1}{\gamma_j}\frac{w_j'G^{-1}w_j}{w_j'q_j} + \frac{q_j'Gq_j-w_j'G^{-1}w_j}{w_j'q_j}\right] .$$

Since

$$|q_j'Gq_j - w_j'G^{-1}w_j| = O(\|w_j - Gq_j\|) \quad ,$$

the last part of the lemma follows from (3.19), (3.7) and Lemmas 1 and 3.

Lemma 5

There are constants $\tau > 2n$ and $\tau^* > 0$ such that

$$\psi_0 \le \tau \quad \text{and} \quad \|x_0-z\| \le \tau^*$$

imply

i) H_j is well-defined and positive definite for $j = 0,1,2,\ldots$

ii) $\psi_j \le \tau_3$ for $j = 0,1,2,\ldots$

iii) $\sum_{j=0}^{\infty} \|x_j-z\| < \infty$.

Proof.

Choose $\lambda = 0.5$ in Lemma 2 and let τ and τ^* be such that

(3.20) $$\tau + 6\delta_1\tau^* \leq \tau_3 \quad \text{and} \quad \tau^* \leq \tau_3^* \ .$$

We will show by induction that then for every j the following statements hold.

(3.21) H_{j+1} is well-defined and positive definite for $i = 0,\ldots,j$

(3.22) $$\|x_{i+1}-z\| \leq 0.5\|x_i-z\|, \quad i = 0,\ldots,j$$

(3.23) $$\psi_{j+1} \leq \psi_0 + 3\delta_1 \sum_{i=0}^{j} \|x_i-z\| \leq \tau_3 \ .$$

Iet $j = 0$. Since (3.20) implies $\tau \leq \tau_3 \leq \tau_2 \leq \tau_1$ and $\tau^* \leq \tau_3^* \leq \tau_2^* \leq \tau_1^*$, it follows from Lemma 3 and the definition of H_0 that H_1 is positive definite. Furthermore, Lemma 2 gives the inequality (3.22) and Lemma 4 in conjunction with (3.6) implies the relation (3.23). Now assume that (3.21) through (3.23) are satisfied for some $j - 1 > 0$. Since (3.22) and (3.23) give the inequalities.

$$\|x_j-z\| < \|x_0-z\| \leq \tau_2^* \quad \text{and} \quad \psi_j \leq \tau_2$$

it follows from Lemma 3 that H_{j+1} is positive definite. Moreover, Lemma 2 implies

$$\|x_{j+1}-z\| \leq 0.5\|x_j-z\| \ .$$

Using Lemma 4 and (3.6) we obtain

$$\begin{aligned}
\psi_{j+1} &\leq \psi_j + 3\delta_1\|x_j-z\| \\
&\leq \psi_0 + 3\delta_1 \sum_{i=0}^{j-1} \|x_i-z\| + 3\delta_1\|x_j-z\| \\
&\leq \psi_0 + 3\delta_1 \sum_{i=0}^{j} (0.5)^i\|x_0-z\| \\
&\leq \psi_0 + 6\delta_1\|x_0-z\| \leq \tau_3 \ .
\end{aligned}$$

Since (3.23) implies that the sum

$$\sum_{j=0}^{\infty} \|x_j-z\|$$

is finite the proof of the lemma is complete.

Using the above results we can now prove that the sequence $\{x_j\}$ generated by the algorithm converges locally and superlinearly to z.

<u>Theorem</u>

Let Assumption 1 be satisfied. For every choice of β_1 and β_2 with $\beta_1 + \beta_2 \neq 0$, there are numbers

$$\delta(\beta_1,\beta_2) \quad \text{and} \quad \delta^*(\beta_1,\beta_2)$$

such that the inequalities

$$\|H_0-G^{-1}\| \leq \delta(\beta_1,\beta_2) \quad \text{and} \quad \|x_0-z\| \leq \delta^*(\beta_1,\beta_2)$$

imply that the sequences $\{x_j\}$ and $\{H_j\}$ in the algorithm are well-defined and have the following properties.

i) H_j is positive definite for all j.

ii) Either $x_j = z$ for some j or

$$\frac{\|x_{j+1}-z\|}{\|x_j-z\|} \to 0 \quad \text{as} \quad j \to \infty$$

$$\sum_{j=0}^{\infty} \left(\frac{\|x_{j+1}-z\|}{\|x_j-z\|}\right)^2 \quad \text{is finite}$$

$\{H_j\}$ and $\{H_j^{-1}\}$ are bounded.

<u>Proof</u>:

It follows immediately from Assumption 1 that there is some neighborhood $U_0(z) \subset U(z)$ such that $x \in U_0(z)$ and $x \neq z$ imply $\nabla F(x) \neq 0$. Choose $\delta^*(\beta_1,\beta_2) \leq \tau^*$ such that $\|x-z\| \leq \delta^*(\beta_1,\beta_2)$ implies $x \in U_0(z)$. Furthermore choose $\delta(\beta_1,\beta_2)$ such that

$$\|H_0-G^{-1}\| \leq \delta(\beta_1,\beta_2) \quad \text{implies} \quad \psi_0 \leq \tau \quad .$$

By (3.7) a $\delta(\beta_1,\beta_2)$ with this property exists for every $\tau > 2n$.

With $\delta(\beta_1,\beta_2)$ and $\delta^*(\beta_1,\beta_2)$ choosen in this way we deduce from Lemma 2 and Lemma 5 that H_j is well-defined and positive definite and $x_j \in U_0(z)$ for every j. Therefore, it follows from Step 1 of the algorithm that $\{x_j\}$ is well-defined. Furthermore $g_j = 0$ if and only if $x_j = z$.

Let $\{x_j\}$ be an infinite sequence. By Lemma 1 and Lemma 5, the sequences $\{H_j\}$ and $\{H_j^{-1}\}$ are bounded.

With $0 < t < 1$ we deduce from (3.6) and Lemma 4 that, for every j,

$$\psi_{j+1} \leq \psi_j + 3\delta_1 \|x_j - z\| + (t-1)\left(\frac{p_j'Gp_j + \rho_j^2 g_j'Gg_j}{\rho_j g_j' p_j} - 2\right) \quad . \tag{3.24}$$

Since by Lemma 1, $\psi_j \geq 2n$ it follows from (3.24) and part iii) of Lemma 5 that

$$\sum_{j=0}^{\infty} \left(\frac{p_j'Gp_j + \rho_j^2 g_j'G^{-1} g_j}{\rho_j g_j' p_j} - 2\right) < \infty \quad ,$$

which by (3.7) implies that

$$\sum_{j=0}^{\infty} \|\rho_j g_j - Gp_j\|^2 < \infty \quad . \tag{3.25}$$

Using (3.12), (3.25), (3.14) and part iii) of Lemma 5 we obtain

$$\sum_{j=0}^{\infty} \left(\frac{\|g_{j+1}\|}{\|g_j\|}\right)^2 < \infty \quad .$$

In view of (3.13) this inequality completes the proof of the theorem.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #1949	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 1 LOCAL AND SUPERLINEAR CONVERGENCE OF A CLASS OF VARIABLE METRIC METHODS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Klaus/Ritter		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 5 - Mathematical Programming and Operations Research
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE 11 Apr ~~il 1~~979
		13. NUMBER OF PAGES 15 12 19 p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 9 Technical summary rept.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

14 MRC-TSR-1949

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Unconstrained minimization, variable metric method, local convergence, superlinear convergence

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper considers a class of variable metric methods for unconstrained minimization problems. It is shown that with a step size of one each member of this class converges locally and superlinearly.

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